GEOMETRY BACKPAPER EXAM (June 2026)

Answer all the questions. Total 40 marks.

(You may use any theorem proved in class, but you must state it clearly.)

Note. All affine spaces below are assumed to be finite-dimensional. Here, we assume V to be a finite-dimensional vector space with $P(V)$ to be projective space over V. Also, we define $P_n(\mathbb{R}) := P(\mathbb{R}^{n+1})$ and $P_n(\mathbb{C}) := P(\mathbb{C}^{n+1})$.

(1) [10 points] Let $\mathscr{A}$ be an affine space of dimension n over $\mathbb{R}$. Let H be an affine hyperplane defined by the equation $\varphi(M) = c$, where $\varphi : \mathscr{A} \to \mathbb{R}$ is a non-constant affine map.

  (i) Show that $\mathscr{A} \setminus H$ is the disjoint union of two open, convex sets.

  (ii) Let A and B be two points in $\mathscr{A}$ not on H. Show that the segment AB intersects H if and only if A and B belong to different connected components.

(2) [8 points] Let $\mathscr{A}$ be an affine space of dimension at least 2 and $\varphi : \mathscr{A} \longrightarrow \mathscr{A}$ be an affine mapping such that the image of any line under φ is a line that is parallel to it. Prove that φ is a translation or a dilatation.

(3) [8 points] Let U be a non-empty open subset on the unit sphere S^2. Prove that there exists no mapping $f : U \longrightarrow \mathbb{R}^2$ that preserves the distances (i.e., $d(x, y) = \|f(x) - f(y)\|$, $\forall x, y \in S^2$, where d is the induced distance on S^2).

(4) [4 points] Prove that the set of all lines in $P_2(\mathbb{R})$ passing through a fixed point $p \in P_2(\mathbb{R})$ is itself a projective line in the dual space.

(5) [7+3 points]

  (i) Let $\varphi : P_1(\mathbb{C}) \longrightarrow P_1(\mathbb{C})$ be a bijective map that preserves cross-ratio of any four points in $P_1(\mathbb{C})$. Then, show that φ is a projective transformation.

  (ii) Let $\varphi : P_n(\mathbb{C}) \longrightarrow P_n(\mathbb{C})$ be a projective transformation and let H_∞ denote the hyperplane at infinity. Suppose that for any two projective lines L_1, L_2 not totally contained in H_∞ such that their affine parts $L_1 \setminus H_\infty$ and $L_2 \setminus H_\infty$ are parallel, their images $\varphi(L_1) \setminus H_\infty$ and $\varphi(L_2) \setminus H_\infty$ are also parallel. Prove that $\varphi(H_\infty) = H_\infty$.